Critical phenomena during a dimensional crossover

1992 J. Phys. A: Math. Gen. 25 101

(http://iopscience.iop.org/0305-4470/25/1/014)

J. Phys. A: Math. Gen. 25 (1992) 101-108. Printed in the UK

Critical phenomena during a dimensional crossover

Denjoe O'Connor† and C R Stephens‡§

† Dublin Institute of Advanced Studies, 10 Burlington Road, Dublin 4, Ireland
‡ Department of Physics, Imperial College, London SW7 2BZ, UK

Received 4 January 1991, in final form 30 August 1991

Abstract. In this paper the critical behaviour of a system undergoing a dimensional crossover is investigated. A renormalization group equation is obtained that interpolates between different dimensions. The susceptibility of an Ising-like system is treated in some detail, in particular an effective critical exponent for the susceptibility is computed, which for the particular crossover examined interpolates between the susceptibility exponents appropriate for a $(4-\varepsilon)$-dimensional and a $(3-\varepsilon)$-dimensional Ising model.

From finite-size scaling arguments we expect [1] that the crossover from the critical behaviour characteristic of one dimension to that of another should have a universal character. Apart from some exact models and some numerical investigations [2] (see [3] for a good review) attempts to investigate this crossover quantitatively have either not been very general or very successful [4]. It is important to rectify this situation for several reasons. Chief amongst these are (i) to provide a possible explanation of some 'anomalous' experimental results in finite-size scaling [5] and (ii) to examine the validity of the finite-size scaling hypothesis [1] itself throughout the dimensional crossover. In this paper we sketch a method of calculating the crossover for an Ising-type system and give explicit results for the susceptibility.

Our approach is to derive a renormalization group equation that corresponds qualitatively to a lattice decimation procedure. One knows that, in a lattice where one of the dimensions is of finite extent, decimation will eventually run out of lattice points to eliminate in the finite direction. Further decimation can only occur in the remaining directions. In continuum language this type of lattice renormalization is described by renormalization group equations. We therefore should look for renormalization group equations that have this qualitative feature of dimensional crossover. The renormalization group must change from that characteristic of one dimension to that of another. From finite-size scaling arguments we expect the relevant parameter governing which side of the crossover we are at to be L/ξ where L is the size of the finite domain and ξ is the correlation length in the system. In this paper we will obtain a renormalization group equation that has the desired features by using an explicitly L-dependent renormalization scheme.

The particular model we will consider for simplicity will be an Ising-like model on a space $S^1 \times R^{3-\varepsilon}$. In this case L is the circumference of the S^1. Intuitively one would expect that as $L/\xi \to \infty$ the model should have a critical behaviour characteristic of $4-\varepsilon$ dimensions; on the other hand as $L/\xi \to 0$ it should have $(3-\varepsilon)$-dimensional

§ Present address: Institute for Theoretical Physics, Princeton plein 5, PO Box 80.006, 3508 TA Utrecht, The Netherlands.

0305-4470/92/010101+08$03.50

behaviour. Of course it is a completely different matter to develop a quantitative scheme within which this intuition may be verified. Most difficulty, as we shall see, is associated with the regime $L/\xi \sim 1$. We start with the canonical Landau-Ginzburg Lagrangian

$$L = \tfrac{1}{2}(\nabla\phi_B)^2 + \tfrac{1}{2}m_B^2\phi_B^2 + \frac{\lambda_B}{4!}\phi_B^4 + \tfrac{1}{2}t_B\phi_B^2. \tag{1}$$

We include the term $\frac{1}{2}t_B\phi_B^2$ and consider it to be in the interaction Lagrangian; this enables us to shift away from the critical point. The analysis of this theory has a long history, in particular in the theory of finite-size scaling (see for example [4]) and in a different guise in finite-temperature field theory (see for instance [6]). The canonical lore on the matter seems to be that one should go ahead and renormalize this theory in the 'standard' manner, e.g. by using minimal subtraction in $4-\varepsilon$ dimensions. The philosophy is that because renormalization is concerned with ultraviolet divergences the same counterterms which suffice to renormalize the bulk theory will renormalize the finite system. However, though the bulk counterterms are sufficient to remove ultraviolet divergences to all orders, in the limit $L/\xi \to 0$ new divergences appear which are not associated with the short-distance behaviour. This can be simply illustrated by calculating to 1-loop the inverse susceptibility of the system [7] which is

$$\chi^{-1} = \tau + \frac{\bar\lambda}{32\pi^2}\tau \ln\frac{\tau}{\kappa^2} + \frac{\bar\lambda}{32\pi^2}\kappa^2 \sum_{n\neq 0}\int_0^\infty \frac{ds}{s^2}\exp\left(-\frac{\tau}{\kappa^2}s\right)\exp\left(-\frac{n^2\pi^2L^2\kappa^2}{4s}\right) \tag{2}$$

where $\tau = T - T_c(\infty)$, $T_c(\infty)$ being the critical temperature of the bulk system, and $\bar\lambda$ is the dimensionless coupling constant. In the limit $L^2\tau \to 0$ the 1-loop correction term is diverging relative to the tree-level term even though we have regulated the ultraviolet regime. One cannot hope, therefore, to get a sensible three-dimensional limit from this approach. Higher loop corrections are even more divergent.

An alternative procedure one might choose to follow starts by Fourier expanding $\phi(x, y)$

$$\phi(x,y) = \frac{1}{L^{1/2}}\sum_{n\in\mathbf{Z}}\phi_n(x)\exp\left(\mathrm{i}\frac{2\pi n y}{L}\right). \tag{3}$$

This leads to an effective Lagrangian, now in terms of renormalized quantities

$$\begin{aligned} L_{\text{eff}} = {} & \tfrac{1}{2}(\nabla\phi_0)^2 + \tfrac{1}{2}m^2\phi_0^2 + \tfrac{1}{2}t\phi_0^2 + \frac{\lambda}{4!L}\phi_0^4 \\ & + \tfrac{1}{2}\sum_{n\neq 0}\left\{\nabla\phi_n\nabla\phi_{-n} + \left(m^2 + t + \frac{4\pi^2n^2}{L^2}\right)\phi_n\phi_{-n}\right\} \\ & + \frac{\lambda}{4!\,L}\sum_{n_1=n_2=n_3=n_4\neq 0}\delta(n_1+n_2+n_3+n_4)\phi_{n_1}\phi_{n_2}\phi_{n_3}\phi_{n_4}. \end{aligned} \tag{4}$$

By an invocation of the Applequist–Carrazone decoupling theorem [8] or some such, one would argue that in the limit $L \to 0$ the $n>0$ modes become very massive and decouple from the theory leaving an effective theory of just the $n=0$ mode. One would then proceed to renormalize this effective theory. Indeed, one would find that the relevant critical exponents were those of a three-dimensional theory. However, having dropped all the $n \neq 0$ modes, and in particular any L dependence, one could never get any information about the higher-dimensional theory. In terms of n the passage from the three-dimensional to the four-dimensional limit is completely non-perturbative. This non-perturbative aspect is the reason why the regime $L^2\tau \sim 1$ is so difficult to treat. Charged with the information that the only true critical point of the system for fixed L is the three-dimensional one, it is natural to choose an approach that emphasizes this. Since (1) emphasizes the bulk theory whereas (4) emphasizes the three-dimensional theory we find (4) a more appropriate starting point. For $L/\xi \to 0$ we expect the $n=0$ sector to dominate. However, we also know that (4) is just (1) rewritten in a different format and consequently that the higher-dimensional behaviour should be recoverable from it. So let us proceed and first calculate the β function for the theory.

In calculating the renormalized coupling constant we will use the following normalization condition for the renormalized four-point coupling of the $n=0$ modes of the Lagrangian (4), $\Gamma^{(4)}_{0000}|_{\mathrm{SP}} = \bar{\lambda}\kappa^{\varepsilon}/L$, where SP denotes the symmetric point. The relationship between the bare and renormalized coupling constants to 1-loop is then

$$\lambda_B = \bar{\lambda}\kappa^{\varepsilon} + \frac{3\bar{\lambda}^2}{2L}\kappa^{2\varepsilon} \sum_{n=-\infty}^{\infty} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^{3-\varepsilon}k}{(2\pi)^{3-\varepsilon}} \frac{1}{(k^2 + 4\pi^2 n^2/L^2 + \kappa^2 x(1-x))^2}. \tag{5}$$

In contrast to what one obtains by minimal subtraction the counterterms in this prescription are L dependent. The β function $\beta \equiv \kappa\, \mathrm{d}\bar{\lambda}/\mathrm{d}\kappa|_{L,\lambda_B}$ arising from this subtraction is

$$\beta(\bar{\lambda}) = -\varepsilon\bar{\lambda} + \frac{3\bar{\lambda}^2}{16\pi^2\kappa L}\Gamma\left(\frac{3+\varepsilon}{2}\right)(4\pi)^{(1+\varepsilon)/2} \times \sum_{n=-\infty}^{\infty} \int_0^1 \frac{x(1-x)\,\mathrm{d}x}{(x(1-x) + 4\pi^2 n^2/\kappa^2 L^2)^{(3+\varepsilon)/2}}. \tag{6}$$

For $L\kappa \to \infty$ we can replace the sum by an integral to find in an ε-expansion

$$\beta(\bar{\lambda}) = -\varepsilon\bar{\lambda} + \frac{3\bar{\lambda}^2}{16\pi^2} + \mathrm{O}(\bar{\lambda}^2\,\mathrm{e}^{-L\kappa}). \tag{7}$$

For $L\kappa \to 0$ only the $n=0$ term is important giving

$$\beta(\bar{\lambda}) = -\varepsilon\bar{\lambda} + \frac{3\bar{\lambda}^2}{16\pi^2\kappa L}\Gamma\left(\frac{3+\varepsilon}{2}\right)(4\pi)^{(1+\varepsilon)/2} \int_0^1 \mathrm{d}x(x(1-x))^{-(1+\varepsilon)/2}. \tag{8}$$

An equivalent but more satisfying expression than equation (8) may be obtained by realizing that as seen in (4) the effective three-dimensional coupling constant is not λ but λ/L, or in terms of dimensionless couplings $\bar{\lambda}/\kappa L$, which we all u. In terms of this effective coupling

$$\beta(u) = -\varepsilon' u + \frac{3u^2}{16\pi^2} + \mathrm{O}(u^2 L^2 \kappa^2) \tag{9}$$

where $\varepsilon' = 1 + \varepsilon$. Thus we see that the β function (6) interpolates in a smooth fashion between the $3-\varepsilon(\equiv 4-\varepsilon')$-dimensional and $(4-\varepsilon)$-dimensional fixed points.

After demonstrating that a subtraction based on normalization conditions gives a satisfactory β function, in contradistinction to straightforward minimal subtraction, we will now show that there is a modified version of minimal subtraction which will work. This simpler procedure we will call generalized minimal subtraction (GMS) after Amit and Goldschmidt [9] who used an analogous method to consider crossover behaviour from an $\mathrm{O}(N)$ to an $\mathrm{O}(M)$ model in fixed dimension at a bicritical point. The essence of the technique is that one should choose counterterms that remove both ultraviolet divergences and those arising from some other relevant limit, e.g. $\tau L^2 \to 0$, as discussed in association with equation (2). In our problem we would like both the

$L \to 0$ and the $L \to \infty$ limits to have sensible finite-renormalized values. With this in mind the relationship between λ_B and $\bar{\lambda}$ in GMS is

$$\lambda_B = \bar{\lambda}\kappa^{\varepsilon'-1} + \frac{3\bar{\lambda}^2}{2L\kappa}\frac{\kappa^{\varepsilon'-1}}{16\pi^2}(4\pi)^{\varepsilon'/2}\Gamma\left(\frac{\varepsilon'}{2}\right)\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2 n^2}{L^2\kappa^2}\right)^{-\varepsilon'/2}. \tag{10}$$

Thus

$$\beta(\bar{\lambda}) = -(\varepsilon'-1)\bar{\lambda} + \frac{3\bar{\lambda}^2}{16\pi^2 L\kappa}(4\pi)^{\varepsilon'/2}\Gamma\left(1+\frac{\varepsilon'}{2}\right)\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2 n^2}{L^2\kappa^2}\right)^{-1-\varepsilon'/2}. \tag{11}$$

For $L\kappa \to \infty$ and $L\kappa \to 0$ we get back equations (7) and (9) respectively. If one uses an ε'-expansion on the last term of equation (11) one obtains the surprisingly simple form

$$\beta(\bar{\lambda}) = -(\varepsilon'-1)\bar{\lambda} + \frac{3\bar{\lambda}^2}{32\pi^2}\coth\frac{\kappa L}{2}. \tag{12}$$

The thing that is significant about these results is that $\beta \equiv \beta(\bar{\lambda}, L\kappa)$, i.e. β is explicitly L-dependent. This has come about because we have used renormalization schemes which are explicitly L-dependent, and this in turn was necessitated by the requirement that the theory have a sensible $L \to 0$ limit. Some insight into the nature and role played by L-dependent renormalization schemes can be gained by examining further the GMS scheme. If one examines equation (10), one sees that the GMS term interpolates between the minimal subtraction schemes appropriate for $(4-\varepsilon')$- and $(4-\varepsilon)$-dimensional systems, yielding leading terms $\sim 1/\varepsilon'$, and $1/\varepsilon$ in the respective limits $L\kappa \to 0$ and $L\kappa \to \infty$.

The running coupling constant $\bar{\lambda}(\rho)$, where ρ is just the standard dilatation parameter, is a solution of the characteristic equation

$$\rho\frac{\mathrm{d}u(\rho)}{\mathrm{d}\rho} = \beta(u(\rho), L\kappa\rho) \tag{13}$$

with the initial condition $u(1) = u$. Solving (13) using equation (12) one finds

$$u^{-1}(\rho) = \rho^{\varepsilon'}u^{-1} - \frac{3L\kappa_0}{32\pi^2}\rho^{\varepsilon'}\int_1^{\rho} x^{-\varepsilon'}\coth\frac{L\kappa_0 x}{2}\,\mathrm{d}x. \tag{14}$$

As $\rho \to 0$ for fixed L, i.e. as we approach the critical point, one can use a small argument expansion of $\coth x$ to find

$$u^{-1}(\rho) = \rho^{\varepsilon'}u^{-1} + \frac{3}{16\pi^2\varepsilon'}(1-\rho^{\varepsilon'}) - \frac{(L\kappa_0)^2}{128\pi^2}\frac{(\rho^2-\rho^{\varepsilon'})}{(2-\varepsilon')} + \mathrm{O}((L\kappa_0)^4) \tag{15}$$

which gives $u(\rho) \to 16\pi^2\varepsilon'/3$ as $\rho \to 0$. For $L\kappa_0 \to \infty$, $\rho \to 0$ but $L\kappa_0\rho \gg 1$ one uses a large argument expansion of $\coth x$ to find, after changing back to $\bar{\lambda}(\rho) = (L\kappa\rho/2)u(\rho)$, that

$$\bar{\lambda}^{-1}(\rho) = \bar{\lambda}^{-1}\rho^{\varepsilon} - \frac{3}{8\pi^2\varepsilon}(1-\rho^{\varepsilon}) + \frac{3\rho^{\varepsilon}}{8\pi^2}\int_1^{\rho} x^{-\varepsilon}\,\mathrm{e}^{-L\kappa_0 x}\,\mathrm{d}x \tag{16}$$

which gives in the limit $\rho \to 0$, $L\kappa_0 \to \infty$, $L\kappa_0\rho \to \infty$ that $\bar{\lambda} \to 8\pi^2\varepsilon/3$. Whether we reach the $(4-\varepsilon)$- or $(4-\varepsilon')$-dimensional fixed point clearly depends on how we treat the $\rho \to 0$ limit. If, for fixed $L\kappa_0$ we let $\rho \to 0$, which physically corresponds to letting $L/\xi \to 0$, then the $(4-\varepsilon')$-dimensional fixed point is reached; however, if $\rho \to 0$ but at the same time $L\kappa_0 \to \infty$ such that $L\kappa_0\rho \to \infty$, the $(4-\varepsilon)$-dimensional fixed point will be reached. This is the conventional regime of finite-size scaling [1, 10]. By solving the renormalization group equation

$$\left(\kappa\frac{\partial}{\partial\kappa}+\beta(u, L\kappa)\frac{\partial}{\partial u}-\frac{N}{2}\gamma_\phi(u, L\kappa)\right)\Gamma^{(N)}_{0\dots0}(k_i, L, u, \kappa)=0 \tag{17}$$

where $\gamma_\phi=\kappa(\partial\ln Z_{\phi_0})/\partial\kappa|_{\lambda_B,L}$ one can show that

$$\Gamma^{(N)}(\rho k_i, L, u, \kappa)$$
$$=\rho^{d-N((d/2)-1)}\exp\left(-\frac{N}{2}\int_1^\rho\frac{\mathrm{d}\rho'}{\rho'}\gamma_\phi(u(\rho'), L\kappa\rho')\right)\Gamma^{(N)}(k_i, L\rho, u(\rho), \kappa) \tag{18}$$

where $u(\rho)$, the solution of the characteristic equation (13), must be inserted in γ_ϕ. Notice that as $\rho\to0$, $L\rho\to0$, in other words the critical region is equivalent to letting $L\to0$.

Equation (18) can now serve as a starting point for calculating crossover functions and effective critical exponents. To do this we need to consider the theory above the critical point. To this end we will set the renormalized mass $m^2=0$ in (4). The parameter $t=T-T_c(L)$, $T_c(L)$ being the critical temperature of the system of size L as distinguished from the bulk critical temperature $T_c(\infty)$, then becomes a measure of how far the system is away from criticality. It is in fact the physically sensible variable with which to quantify the dimensional crossover. The relationship between t and t_B is $t_B=Z_{\phi^2}t$. We demand that Z_{ϕ^2} have a sensible $L\to0$ limit so once again an L-dependent renormalization scheme must be used, either in the guise of a suitable normalization condition or GMS. Above $T_c(L)$ the appropriate renormalization group equation is

$$\left(\kappa\frac{\partial}{\partial\kappa}+\beta(u, L\kappa)\frac{\partial}{\partial u}-\frac{N}{2}\gamma_\phi(u, L\kappa)+\gamma_{\phi^2}(u, L\kappa)t\frac{\partial}{\partial t}\right)\Gamma^{(N)}=0 \tag{19}$$

where $\gamma_{\phi^2}=-\kappa(\partial\ln Z_{\phi^2})/\partial\kappa|_{L,\lambda_B}$. One characteristic equation of (19) is (13), the equation for the running coupling constant, another

$$\kappa\frac{\mathrm{d}\ln t}{\mathrm{d}\kappa}=\gamma_{\phi^2}(u, L\kappa) \tag{20}$$

leads to a definition of a running temperature $t(\rho)$ via

$$\rho\frac{\mathrm{d}\ln t(\rho)}{\mathrm{d}\rho}=\gamma_{\phi^2}(u(\rho), L\kappa\rho)-2 \tag{21}$$

with the initial condition $t(1)=t$. The solution of this equation is

$$t(\rho)=\frac{t}{\rho^2}\exp\left(\int_1^\rho\frac{\mathrm{d}\rho'}{\rho'}\gamma_{\phi^2}(u(\rho'), L\kappa\rho')\right). \tag{22}$$

One can now use equation (19) to find the relationship between the two-point function $\Gamma^{(2)}_{00}(0, t, L, u)$, which is just the inverse susceptibility χ^{-1}, at two different values of the dilatation parameter; $\rho=1$ and ρ say.

$$\chi^{-1}(t, L, u)=\rho^2\exp\left(-\int_1^\rho\frac{\mathrm{d}\rho'}{\rho'}\gamma_\phi(u(\rho'), L\kappa\rho')\right)\chi^{-1}(t(\rho), L\rho, u(\rho)). \tag{23}$$

As ρ is an arbitrary parameter we can choose it so that $t(\rho)$ is out of the critical region, e.g. $t(\rho)=\kappa^2$. When substituted back into equation (22) this gives us an implicit equation for $\rho\equiv\rho(t, L, \kappa)$. Carrying out the above analysis to 1-loop one finds

$$\chi^{-1}=t+\frac{\bar{\lambda}}{2L}\kappa^{\varepsilon'-1}\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^{4-\varepsilon'}k}{(2\pi)^{4-\varepsilon'}}\frac{1}{k^2+t+4\pi^2n^2/L^2}+c_1+c_2t \tag{24}$$

where c_1 is determined by mass renormalization and c_2 from composite operator renormalization. Using the normalization conditions $\chi^{-1}(0, L, \kappa)=0$ and $\chi^{-1}(\kappa^2, L, \kappa)=\kappa^2$ and dimensional analysis one finds after using equation (23) that

$$\chi^{-1}\left(\frac{t}{\kappa^2}, L\kappa, u\right)=\rho^2 \tag{25}$$

where one must use $\rho\equiv\rho(t)$.

One can define an effective critical exponent γ_{eff} for the susceptibility as

$$\gamma_{\mathrm{eff}}=\frac{\mathrm{d}\ln\chi^{-1}}{\mathrm{d}\ln t}. \tag{26}$$

Therefore

$$\gamma_{\mathrm{eff}}=\frac{\mathrm{d}\ln\rho^2}{\mathrm{d}\ln t}. \tag{27}$$

Using equation (22) and the condition $t(\rho)=\kappa^2$ one obtains

$$\frac{\mathrm{d}\ln\rho^2}{\mathrm{d}\ln t}=1+\tfrac{1}{2}\gamma_{\phi^2}\left(u\left(\frac{t^{1/2}}{\kappa}, Lt^{1/2}\right), Lt^{1/2}\right). \tag{28}$$

An explicit computation of γ_{ϕ^2} yields

$$\begin{aligned}\gamma_{\phi^2}=-\frac{\Gamma(\varepsilon'/2)}{16\pi^2}(4\pi)^{\varepsilon'/2}\sum_{n=-\infty}^{\infty}&\left(\frac{t(\partial u/\partial t)}{(1-\varepsilon'/2)}\left[\left(1+\frac{4\pi^2n^2}{tL^2}\right)^{1-(\varepsilon'/2)}-\left(\frac{4\pi^2n^2}{tL^2}\right)^{1-(\varepsilon'/2)}\right]\right.\\&\left.+u\frac{4\pi^2n^2}{tL^2}\left[\left(1+\frac{4\pi^2n^2}{tL^2}\right)^{-\varepsilon'/2}-\left(\frac{4\pi^2n^2}{tL^2}\right)^{-\varepsilon'/2}\right]\right).\end{aligned} \tag{29}$$

It is not difficult to check that despite appearances this function is finite. The reader might despair as to whether such a formal looking expression has any utility whatsoever as an expression leading to a crossover function. The reason we have left it in such a formal state is that there are several approximation schemes which can be employed in its evaluation. For example one may employ an ε'- or an ε-expansion throughout. Alternatively one could set $\varepsilon'=1$ thereby working in fixed dimension. One may also exploit different approximation schemes in different regimes. For instance, in the regime $tL^2\gg 1$ one might expect an ε-expansion to be more appropriate, whilst for $tL^2\ll 1$ an ε'-expansion would be better. It is important to realize, though, that the true expansion parameter through the entire crossover is the running coupling itself, not ε or ε'; the latter cannot be treated as small parameters throughout the crossover. If for $tL^2\to\infty$ an ε-expansion is used one finds

$$\gamma_{\mathrm{eff}}=1+\frac{\varepsilon}{6}+\mathrm{O}\left(\frac{\varepsilon}{tL^2}\right). \tag{30}$$

On the other hand if for $tL^2\to 0$ an ε'-expansion is used one obtains

$$\gamma_{\mathrm{eff}}=1+\frac{\varepsilon'}{6}+\mathrm{O}(\varepsilon' L^2 t). \tag{31}$$

In the same way that we used an ε' expansion to generate simple expressions from (11) one can derive from (29) the following fairly simple form

$$\gamma_{\text{eff}} = 1 - \frac{u(t/\kappa^2)}{32\pi^2} \sum_{n=-\infty}^{\infty} \left(\frac{4\pi^2 n^2}{tL^2} \ln\left(1 + \frac{tL^2}{4\pi^2 n^2}\right) - 1 \right) \tag{32}$$

where $u(t/\kappa^2)$ must be substituted in from the β-function expressions. It seems quite remarkable that one can obtain such a simple functional form with which to describe the entire crossover regime.

Let us briefly comment on the validity of these results. For $tL^2 \gg 1$ if one performs an ε-expansion one will end up with an answer which has the same asymptotic validity as the conventional 'ε-expansion' in $4-\varepsilon$ dimensions. On the other hand for $tL^2 \ll 1$ if one performs an ε'-expansion one will end up with an expansion in ε' which has the same asymptotic validity as the conventional 'ε-expansion' in $4-\varepsilon'$ dimensions. For $S^1 \times R^2$ we would end up with expressions in the limits $tL^2 \to \infty$ and $tL^2 \to 0$ which were those of a three-dimensional and a two-dimensional system as calculated using standard 'ε-expansions', i.e. in the former $\varepsilon = 1$ and in the latter $\varepsilon = 2$. If we are concerned with the entire crossover then one is relying on the fact that $u(t/\kappa^2, tL^2)$ is small throughout the crossover. This will not be true in general; in this case one would wish to work to multiloop order and perform a Borel summation. In principle this can be done using the methods herein, the only real difficulty being one of computational facility. One would still expect low orders in perturbation theory to give qualitatively correct physics, however.

It is evident from the above discussion that one can go on to examine the other critical exponents and thermodynamic functions to obtain their crossover behaviour. It is also clear that the methodology is much more general than the restricted example considered here. The main insight we have used in the above is to recognize the importance of explicitly taking into account in the renormalization the important fluctuations with momenta $\leqslant 1/L$. This yields a simple adjustment of the counterterms thereby making them explicitly L-dependent. Nevertheless such a simple adjustment leads to a powerful new technique for examining the critical behaviour of systems undergoing a dimensional crossover. The main applications we have in mind at the moment, as stated previously, are to explain the 'anomalous' experimental results in finite-size scaling, and to examine the validity of the finite-size scaling hypothesis throughout the dimensional crossover. Current experimental results associated with finite-size scaling come from studies of liquid He^4. To investigate such a system our results need to be extended to a consideration of the XY model crossing over from three dimensions to two. Such a calculation employing the general strategy outlined here is perfectly feasible. One complication, however, will be the treatment of vortices as these will play an important role as the two-dimensional limit is approached. This would be an interesting calculation not only because of the experimental interest but also because it would illuminate the crossover between an ordinary order/disorder transition and a topological one. As far as experiment is concerned, however, the reader should note that the present formalism is applicable to many other crossovers of interest. For instance, for three-dimensional uniaxial dipolar ferromagnets one can investigate the entire crossover between three-dimensional and quasi-four-dimensional critical behaviour [11].

As well as comparison with experiment it would also be useful to make a thorough comparison with the spherical model where exact calculations can be performed. Here, though, we are considering the situation where both the bulk and finite systems exhibit critical behaviour. For the spherical model this would restrict attention to the crossover between four and three dimensions; however, in such a situation there are apparent logarithmic violations of finite-size scaling. A crucial ingredient in the relationship between the present formalism and the aforementioned are the corrections to scaling. *In our methodology by finding the true L-dependent fixed point of the system we* ensure that corrections to scaling are small. If one expanded around the bulk fixed point one would find very large corrections to scaling when the secondary fixed point played an important role. It just so happens that because the spherical model is exact these corrections to scaling can be computed exactly even when large. In general this will not be so; in such cases our formalism will be superior.

Acknowledgments

One of us (CRS) would like to thank Dr J Soto for useful discussions and wishes to thank the Institutes of Theoretical Physics at the University of Bern and the University of Vienna for hospitality, where some of this work was carried out.

Note added in proof. After the acceptance of this paper we learned of the related work [12].

References

[1] Fisher M E and Barber M N 1972 *Phys. Rev. Lett.* **28** 1516
[2] Barber M N and Fisher M N 1973 *Ann. Phys.* **77** 1
Singh S and Pathria R K 1985 *Phys. Rev.* B **31** 4483
[3] Barber M N 1984 *Phase Transitions and Critical Phenomena* vol VIII; ed C Domb and J Lebowitz (New York: Academic)
[4] Nemirovsky A M and Freed K F 1985 *J. Phys. A: Math. Gen.* **18** L319; 1986 *Nucl. Phys.* B **270** [FS16] 423; 1986 *J. Phys. A: Math. Gen.* **19** 591
Rudnick J, Guo H and Jasnow D 1985 *J. Stat. Phys.* **41** 353
Brezin E and Zinn-Justin J 1985 *Nucl. Phys.* B **257** [FS14] 867
[5] Rhee I, Gasparini F M and Bishop D J 1989 *Phys. Rev. Lett.* **63** 410
Gasparini F M, Agnolet G and Reppy J D 1984 *Phys. Rev.* B **29** 138
[6] Ginsparg P 1980 *Nucl. Phys.* B **170** [FS1] 388
Dolan L and Jackiw R 1974 *Phys. Rev.* D **9** 3320
[7] O'Connor D J, Stephens C R and Hu B L 1989 *Ann. Phys.* **190** 310
[8] Applequist T and Carazzone J 1975 *Phys. Rev.* D **11** 2856
[9] Amit D J and Goldschmidt Y Y 1978 *Ann. Phys.* **114** 356
[10] Brezin E 1982 *J. Physique* **43** 23
[11] Stephens C R 1992 *J. Magnet. Magnet. Mater.* in press
Frey E and Schwabl F 1991 *Phys. Rev.* B **43** 833
[12] Schmeltzer D 1985 *Phys. Rev.* B **32** 7512